ISRAEL JOURNAL OF MATHEMATICS, Vol. 36, Nos. 3–4, 1980

ORBITS OF EUCLIDEAN FRAMES UNDER DISCRETE LINEAR GROUPS

BY

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ABSTRACT

We obtain certain sufficient conditions for the orbit of a (euclidean) p-frame over a vector space V, $p < \dim V$, under the action of a discrete subgroup of $GL(V)$, to be dense in the corresponding orbit of a Lie subgroup of $GL(V)$. Using the result we classify the p-frames whose orbits under $SL(n, \mathbf{Z})$ are dense in the space of p-frames and deduce, in turn, a classification of dense orbits of certain horospherical flows. A similar result is obtained for $Sp(2n, \mathbf{Z})$ for $p \leqq n$.

Introduction

In [3], J. S. Dani proved that for $\Gamma = SL(n, \mathbf{Z})$ or $Sp(2n, \mathbf{Z})$ and v in $V = \mathbf{R}^n$ or $\mathbf{R}^{2n}$ respectively, the orbit Γv is dense in V if and only if v is irrational (i.e. not a scalar multiple of a rational vector). Earlier, Greenberg [7] had shown that for *uniform* lattices Γ in certain classical groups (e.g. $SL(n)$, $Sp(2n)$), Γ-orbits of all non-zero vectors are dense in the ambient vector space. A generalisation of Greenberg's result arises as follows: If G is a semisimple Lie group, Γ a lattice in G and U a horospherical subgroup acting ergodically on G/Γ then every U-orbit on G/Γ is dense. This result was proved independently by several authors including W. A. Veech [13] and also in a paper (cf. [5], proposition 4.6) of the first-named author. Veech also observed that as a consequence of the above result if Γ is a uniform lattice in $SL(n, \mathbf{R})$ and $\mathbf{v}$ is a (euclidean) p-frame then $\Gamma\mathbf{v}$ is dense in the space of p-frames, if $p \leqq n-1$. This led us to consider the density of orbits of p-frames under non-uniform lattices. In the first two sections, we derive sufficient conditions for the orbit of a p-frame under a lattice, not necessarily uniform, to be dense in the orbit under the corresponding Lie group (cf. Theorem 2.1). In §3, we deduce that if $\Gamma = SL(n, \mathbf{Z})$ and $\mathbf{v}$ is a p-frame over $\mathbf{R}^n$ with $p \leqq n-1$, the Γ-orbit of $\mathbf{v}$ is dense in the space of p-frames over $\mathbf{R}^n$ if and only if $\mathbf{v}$ is "irrational" (that is, the p-dimensional subspace "generated" by

Received June 6, 1979 and in revised form March 13, 1980

$\mathbf{v}$ contains no non-zero rational vector). A similar result is proved for $\mathrm{Sp}(2n, \mathbf{Z})$ with $p \leqq n$ (cf. Theorem 3.5).

In §4, we apply Theorem 3.5 to study the density of orbits of certain horospherical subgroups. Let $G = \mathrm{SL}(n, \mathbf{R})$ and $\Gamma = \mathrm{SL}(n, \mathbf{Z})$. Let P be a maximum dimensional parabolic subgroup of G and U be the unipotent radical of P. Then U is a horospherical subgroup of G. We deduce that for $x \in G$ the U-orbit of $x\Gamma$ is dense in G/Γ if and only if $x^{-1}Ux$ is not contained in any maximum dimensional parabolic subgroup defined over $\mathbf{Q}$ (cf. Corollary 4.1). Lastly we also show that the closure of any orbit of U coincides with an orbit of a subgroup H containing U (cf. Corollary 4.3).

It may be noted that our techniques are largely motivated by [3] and [12].

§1. Notations and preliminaries

(a) *Frame spaces.* Let V be a finite dimensional $\mathbf{R}$-vector space equipped with euclidean topology and n be the dimension of V. Let $p \leqq n-1$. A euclidean *p-frame* or a *p-frame over V* is an (ordered) p-tuple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p)$ where $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$ are linearly independent elements of V. For a p-frame $\mathbf{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p)$ we shall denote by $\langle \mathbf{v} \rangle$, or $\langle \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p \rangle$, the subspace generated by $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$. The set of p-frames may be viewed as a subset of $W = V^p = V \times V \times \cdots \times V$ (p copies) and we shall consider it to be equipped with the subspace topology.

Let $\mathrm{GL}(V)$ be the (topological) group of invertible linear transformations of V and G be a closed subgroup of $\mathrm{GL}(V)$. The natural action of G on V induces an action on V^p which restricts to an action on the space of p-frames over V. If $g \in G$ and $\mathbf{v}$ is a p-frame, the product under the action shall be denoted by $g\mathbf{v}$.

(b) *Lie groups and their subgroups.* Let G be a (connected) Lie group and let $g \in G$. The subgroup

$$U = \{u \in G \mid g^j u g^{-j} \to e \text{ as } j \to +\infty\},$$

where e is the identity element in G, is called the *contracting horospherical subgroup* associated to g. The contracting horospherical subgroup associated to g^{-1} is called the *expanding horospherical subgroup* associated to g. Let $\mathfrak{G}$ be the Lie algebra corresponding to G. Let $\mathfrak{G} = \Sigma_{\alpha \in \Lambda} \mathfrak{G}^\alpha$ (direct-sum) be the decomposition into generalised eigenspaces corresponding to $\mathrm{Ad}\, g$; i.e. Λ is a suitable indexing set and for $\alpha \in \Lambda$ there exists an irreducible polynomial f_α over $\mathbf{R}$ such that

$$\mathfrak{G}^\alpha = \{X \in \mathfrak{G} \mid f_\alpha^r(\mathrm{Ad}\, g)X = 0 \text{ for some } r > 0\}.$$

For each α, $\mathfrak{G}^\alpha$ is $\operatorname{Ad} g$-invariant and all the eigenvalues of $\operatorname{Ad} g/\mathfrak{G}^\alpha$ have the same absolute value, say λ_α. Set

$$\mathfrak{u} = \sum_{\lambda_\alpha > 1} \mathfrak{G}^\alpha \quad \text{and} \quad \mathfrak{p} = \sum_{\lambda_\alpha \leqq 1} \mathfrak{G}^\alpha.$$

Then $\mathfrak{u}$ and $\mathfrak{p}$ are Lie subalgebras of $\mathfrak{G}$. Let U and P denote the analytic subgroups of G corresponding to $\mathfrak{u}$ and $\mathfrak{p}$ respectively. It is easy to verify that U is the expanding horospherical subgroup associated to g (cf. [5], proposition 4.1). It may be deduced from this that if G is a linear group then a horospherical subgroup (contracting or expanding) consists of unipotent elements. A partial converse is also true; if G is a reductive (completely reducible) linear group then a subgroup U of G is horospherical if and only if U is the unipotent radical of a parabolic subgroup of G.

The subgroup P as defined above is called the *stable* subgroup associated to g. Consider the map $M : P \times U \to G$ defined by $(x, y) \mapsto x \cdot y$ where $x \in P$ and $y \in U$. Since M is analytic and the tangent map at e, (the identity), is obviously an isomorphism, by inverse function theorem we obtain the following:

1.1. Lemma. *There exists a neighbourhood P_1 of e in P and a neighbourhood U_1 of e in U such that the restriction of M to $P_1 \times U_1$ is a diffeomorphism of $P_1 \times U_1$ onto a neighbourhood Ω of e in G.*

1.2. Lemma. *Let $g \in G$ be such that* $\operatorname{Ad} g$ is semisimple (diagonalisable over C). *Let $\{p_k\}$ be a sequence in the stable subgroup P of g such that $p_k \to e$ (as $k \to \infty$). Let $\{n_k\}$ be a sequence in* N. *Then $g^{n_k} p_k g^{-n_k} \to e$.*

Proof. Let $\mathfrak{G}^\alpha$ and λ_α, $\alpha \in \Lambda$ be as before. Since $\operatorname{Ad} g$ is now assumed to be semisimple $\lambda_\alpha^{-1} \operatorname{Ad} g$ is a rotation of $\mathfrak{G}^\alpha$. Hence there exists a norm $\|\cdot\|$ on $\mathfrak{G}$ such that for $X \in \mathfrak{G}^\alpha$ $\|(\operatorname{Ad} g)X\| = \lambda_\alpha \|X\|$. Without loss of generality, we may assume that $p_k = \exp X_k$ where $X_k \in \mathfrak{p}$ (the Lie subalgebra corresponding to P) and $X_k \to 0$. Then $g^{n_k} p_k g^{-n_k} = g^{n_k} (\exp X_k) g^{-n_k} = \exp((\operatorname{Ad} g)^{n_k} X_k)$. But since $X_k \in \mathfrak{p}$, $\|(\operatorname{Ad} g)^{n_k} X_k\| \leqq \|X_k\| \to 0$. Hence $g^{n_k} p_k g^{-n_k} \to e$.

(c) *Semisimple Lie groups.* Let G be a (connected) semisimple Lie group which admits no non-trivial compact factor. Let Γ be a lattice in G; i.e. Γ is a discrete subgroup and G/Γ admits a finite Borel measure μ which is invariant under the action of G (on the left). A subgroup L of G is said to be *totally non-compact relative to* Γ if for any homomorphism $\rho : G \to S$, onto a Lie group S of positive dimension, such that $\rho(\Gamma)$ is closed, the closure of $\rho(L)$ in S is non-compact.

1.3. PROPOSITION. *In the notation as above if L is a subgroup of G which is totally non-compact relative to Γ then the action of L on G/Γ is ergodic; that is, if E is a Borel subset of G/Γ such that $\mu(E \triangle xE) = 0$ for all $x \in L$ then either $\mu(E) = 0$ or $\mu(G/\Gamma - E) = 0$.*

PROOF. Let $G_1, G_2, \cdots, G_r$ be the connected simple normal (closed) subgroups of G. Let G^* be the adjoint group of G and G_i^*, $i = 1, 2, \cdots, r$ the images of G_i under the adjoint homomorphism $\mathrm{Ad}: G \to G^*$. Then $G^* = \Pi G_i^*$. Let p_i denote the ith projection. Let T be the unitary representation of G on $\mathcal{H} = \mathcal{L}^2(G/\Gamma, \mu)$ corresponding to the G-action on G/Γ. To prove ergodicity of the L-action it is enough to show that every $T(L)$-invariant function in $\mathcal{H}$ is constant μ-a.e. If i is an index such that the closure of $p_i(\mathrm{Ad}\, L)$ in G_i^* is noncompact then by C. C. Moore's ergodicity theorem (cf. [8] and also [4] §4) every $T(L)$-invariant element of $\mathcal{H}$ is $T(G_i)$-invariant. Let G' be the product of all G_i satisfying the above condition. Let F be the connected component of the identity in $\mathrm{Cl}(G'\Gamma)$; ("Cl" stands for "closure of"). In view of Borel's density theorem F is a normal subgroup of G. Let $\rho: G \to G/F$ be the quotient homomorphism. Then evidently $\rho(\Gamma)$ is closed and $\mathrm{Cl}\,\rho(L)$ is compact. Since L is totally non-compact relative to Γ, G/F must be trivial. Hence $F = G$ and consequently $G'\Gamma$ is dense in G. Since G' is normal in G (by a simple argument involving the representation of G on the space of locally integrable functions on G) this implies that the action of G' on G/Γ is ergodic (cf. for instance [4], corollary 3.2 for an idea of the proof). Hence every $T(G')$-invariant element of $\mathcal{H}$ is constant μ a.e. Hence every $T(L)$-invariant element is constant μ a.e.

REMARK. In Proposition 1.3 as also in Theorem 2.1, we may drop the assumption that G has no non-trivial compact factors provided Γ is a lattice such that under any homomorphism of G onto a compact group the image of Γ is dense. However we choose to ignore this to avoid complicating the statement of Theorem 2.1.

(d) *Dense orbits.* The following result is well-known (and also easy to prove).

1.4. PROPOSITION. *Let X be a second countable topological space. Let μ be a Borel measure on X such that $\mu(\Omega) > 0$ for every non-empty open set Ω of X. Let L be a topological group acting continuously on X and suppose that the action is ergodic with respect to μ. Then L-orbits of μ-almost all points are dense in X. If φ is an ergodic homeomorphism of X (i.e. the action of $\{\varphi^j \mid j \in \mathbf{Z}\}$ is ergodic) with respect to μ then for μ-almost all points x, $\{\varphi^j x \mid j \in \mathbf{N}\}$ is dense in X.*

1.5. Proposition. *Let G be a Lie group and Γ be a lattice in G. Let $g \in G$ be such that* Ad g *is semisimple and the action of g on G/Γ is ergodic (with respect to the G-invariant measure). Let U be the expanding horospherical subgroup associated to g. Let H be the subgroup generated by U and g. Then every H-orbit on G/Γ is dense.*

Proof. Let $x, y \in G/\Gamma$ be given. Since the action of g is ergodic, in view of Proposition 1.4 there exists a sequence $\{g_k\}$ in G, $g_k \to e$, the identity, such that for each $k \in \mathbf{N}$, $\{g^j(g_kx) \mid j \in \mathbf{N}\}$ is dense in G/Γ. Hence there exists a sequence $\{n_k\}$ in $\mathbf{N}$ such that $g^{n_k}g_kx \to y$.

Now let P be the stable subgroup associated to g and let P_1 and U_1 be the neighbourhoods of the identity in P and U respectively such that the contention of Lemma 1.1 holds. Then there exist sequences $\{p_k\}$ in P and $\{u_k\}$ in U such that as $k \to \infty$, $p_k \to e$, $u_k \to e$ and $g_k = p_ku_k$ for all sufficiently large k. Since $n_k \geqq 0$ and $p_k \to e$ by Lemma 1.2 $g^{n_k}p_kg^{-n_k} \to e$ as $k \to \infty$. Hence

$$g^{n_k}u_kx = (g^{n_k}p_k^{-1}g^{-n_k})(g^{n_k}g_kx) \to e \cdot y = y.$$

Since $g^{n_k}u_k \in H$ and x and y are arbitrary, we conclude that every H-orbit on G/Γ is dense.

The basic idea involved in the proof of the above lemma is due to G. D. Mostow. The authors are indebted to G. Prasad for a discussion pertaining to the proof.

§2. Dense orbits of lattices

The aim of this section is to prove the following result.

2.1. Theorem. *Let G be a (connected) semisimple Lie subgroup of* $\mathrm{GL}(V)$ *where V is a finite dimensional $\mathbf{R}$-vector space of dimension n. Assume that G admits no non-trivial compact factor. Let Γ be a lattice in G. Let $p \leqq n-1$ and $\mathbf{v} = (v_1, v_2, \cdots, v_p)$ be a p-frame over V satisfying the following conditions:*

(i) *There exists a one-parameter subgroup $\{\varphi_t\}$ of G such that* (a) *$\{\varphi_t\}$ is totally non-compact relative to Γ and* (b) *there exist $v_{p+1}, v_{p+2}, \cdots, v_n \in V$ such that $\{v_1, v_2, \cdots, v_n\}$ is a basis of V and for all $t \in \mathbf{R}$, $\varphi_t\mathbf{v} = e^t\mathbf{v}$ and $\varphi_tv_i = e^{s_it}v_i$ where $s_i < 1$ for all $i = p+1, \cdots, n$.*

(ii) *$G\mathbf{v}$ is closed in the space of p-frames.*

(iii) *There exist sequences $\{\gamma_j\}$ in Γ and $\{\lambda_j\}$ in $\mathbf{R}_+^*$ such that as $j \to \infty$, $\gamma_j\mathbf{v} \to 0$ and $\lambda_j\gamma_j\mathbf{v} \to \mathbf{w}$ where $\mathbf{w}$ is a p-frame over V.*

Then $\Gamma\mathbf{v}$ is dense in $G\mathbf{v}$.

REMARK. We note that conditions (i) and (ii) depend only on the G-orbit of $\mathbf{v}$. Indeed to obtain condition (i) for $g\mathbf{v}$ for some $g \in G$ one only needs to consider the one-parameter subgroup $\{g\varphi_t g^{-1}\}$. For condition (ii) the assertion is obvious. On the other hand it turns out that, in general, condition (iii) depends on $\mathbf{v}$ within the G-orbit.

EXAMPLES. (a) Let $G = \mathrm{SL}(V)$ and $p \leqq n-1$. Let $\mathbf{v} = (v_1, v_2, \cdots, v_p)$ be a p-frame. Let $v_{p+1}, v_{p+2}, \cdots, v_n \in V$ be such that $\{v_1, v_2, \cdots, v_n\}$ is a basis of V. Let φ_t, $t \in \mathbf{R}$ be the transformation defined by $\varphi_t v_i = e^t v_i$ for $i \leqq p$ and $\varphi_t v_i = e^{-st} v_i$ for $p+1 \leqq i \leqq n$ where $s = p/(n-p)$. Since $\mathrm{SL}(V)$ is an almost simple Lie group with finite center and $\{\varphi_t\}$ is a closed non-compact subgroup, it is totally non-compact (relative to any lattice). Hence condition (i) is satisfied. Condition (ii) follows since $G\mathbf{v}$ obviously coincides with the space of p-frames.

(b) Let G be the group $\mathrm{Sp}(2n, \mathbf{R})$; i.e. $G = \{X \in \mathrm{SL}(2n, \mathbf{R}),\ XJ_n{}^tX = J_n\}$ where ${}^tX$ denotes the transpose of X and

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

where E_n denotes the (n, n) identity matrix. Consider the natural action of G on $\mathbf{R}^{2n}$. Let $e_i = {}^t(0, 0, \cdots, 0, 1, 0, \cdots, 0)$, $1 \leqq i \leqq 2n$, where the entry 1 is at the ith place. Let $\mathbf{v}$ be the p-frame $(e_1, e_2, \cdots, e_p)$ where $p \leqq n$. Let $\varphi_t, t \in \mathbf{R}$, be the diagonal matrix defined by $\varphi_t e_i = e^t e_i$ and $\varphi_t e_{n+i} = e^{-t} e_{n+i}$ for $i \leqq p$ and $\varphi_t e_i = e_i$ and $\varphi_t e_{n+i} = e_{n+i}$ for $p+1 \leqq i \leqq n$. Then $\varphi_t \in \mathrm{Sp}(2n, \mathbf{R})$ and as in the previous example, condition (i) is verified.

A p-frame $(w_1, \cdots, w_p)$ over $\mathbf{R}^{2n}$ is called *symplectic* if ${}^tw_i J_n w_j = 0$ for $1 \leqq i, j \leqq p$. It is well-known that a p-frame $(w_1, \cdots, w_p)$ is symplectic if and only if $w_1, \cdots, w_p$ form the first p columns of an element of $G = \mathrm{Sp}(2n, \mathbf{R})$ (cf. [1], theorem 3.8). Thus a p-frame $\mathbf{w} = (w_1, \cdots, w_p)$ is in $G\mathbf{v}$ if and only if $\mathbf{w}$ is symplectic. Therefore condition (ii) in Theorem 2.1 is evidently satisfied.

In the next section, we shall identify the frames involved in Example (a) (respectively Example (b)) which satisfy condition (iii) for the case when $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ (respectively when $\Gamma = \mathrm{Sp}(2n, \mathbf{Z})$). In both the cases we shall also conclude the converse statement that orbits not satisfying condition (iii) are not dense.

The proof of Theorem 2.1 follows through a sequence of Lemmas. To begin with, we shall let $\mathbf{v} = (v_1, v_2, \cdots, v_p)$ be an arbitrary p-frame. The conditions of the theorem will be brought into play as the proof develops. Set $L = \{g \in G \mid g\mathbf{v} = \mathbf{v}\}$, $H = \{g \in G \mid \exists \alpha \in \mathbf{R}$ such that $g\mathbf{v} = \alpha\mathbf{v}\}$ and

$$P = \{g \in G \mid g\langle v_1, v_2, \cdots, v_p\rangle = \langle v_1, v_2, \cdots, v_p\rangle\}.$$

Then L, H and P are closed subgroups of G and $L \subset H \subset P$. We also note that L is a normal subgroup of P. Indeed, the map $\rho : P \to \mathrm{GL}(\langle v_1, v_2, \cdots, v_p \rangle)$ defined by restriction is a homomorphism with L as its kernel. Let U be the expanding horospherical subgroup associated to φ_1. A straight-forward computation using the definition of U, shows that $\mathbf{v}$ is fixed under the action of U. Therefore U is contained in L.

Let $W = V^p$ and $\mathbf{P} = \mathbf{P}(W)$ be the corresponding projective space. Let $\pi : W - (0) \to \mathbf{P}$ be the natural projection. The subgroups L and H are the isotropy subgroups of $\mathbf{v}$ and $\pi(\mathbf{v})$ respectively under the G-actions on W and $\mathbf{P}$ respectively. Let $\alpha : G/L \to W$ and $\beta : G/H \to \mathbf{P}$ be the maps defined by $\alpha(gL) = g\mathbf{v}$ and $\beta(gH) = \pi(g\mathbf{v})$ respectively. It is well-known that the images of α and β are locally compact subsets of the respective spaces and that consequently α and β are homeomorphisms onto their images. Also observe that α and β are equivariant with respect to the G-actions. Hence the Γ-orbit of an element x in G/L (resp. y in G/H) is dense in G/L (resp. G/H) if and only if the Γ-orbit of $\mathbf{v} = \alpha(x)$ (resp. $\pi(\mathbf{v}) = \beta(x)$) is dense in $G\mathbf{v}$ (resp. $\pi(G\mathbf{v})$).

2.2. Lemma. *Let $\mathbf{v}$ and $\mathbf{v}'$ be two p-frames over V such that $\langle \mathbf{v} \rangle = \langle \mathbf{v}' \rangle$. Then $\Gamma\mathbf{v}$ is dense in $G\mathbf{v}$ if and only if $\Gamma\mathbf{v}'$ is dense in $G\mathbf{v}'$.*

Proof. Viewing $\mathbf{v}$ and $\mathbf{v}'$ as (n, p) matrices (with respect to a fixed basis) we see that $\mathbf{v}' = \mathbf{v} \cdot A$ for some $A \in \mathrm{GL}(p, \mathbf{R})$. Hence $\Gamma\mathbf{v}' = (\Gamma\mathbf{v})A$ is dense in $G\mathbf{v}' = (G\mathbf{v})A$ if and only if $\Gamma\mathbf{v}$ is dense in $G\mathbf{v}$.

2.3. Lemma. *Let $\mathbf{v}$ be a p-frame satisfying condition* (i) *of Theorem* 2.1 *and let $\{\varphi_t\}$ be the one-parameter subgroup involved in the condition. Assume that there exists a Cartan subgroup A (centraliser of a Cartan subalgebra) containing $\{\varphi_t\}$ such that $A\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle$ and $A \cap \Gamma$ is a lattice in A. Then $\Gamma\mathbf{v}$ is dense in $G\mathbf{v}$.*

Proof. In the notation preceding Lemma 2.2, we only need to show that $\Gamma L/L$ is dense in G/L. Recall that L contains the (expanding) horospherical subgroup U corresponding to $\{\varphi_t\}$. Since $\{\varphi_t\}$ is totally non-compact relative to Γ, it follows that U is also totally non-compact relative to Γ. Therefore so is L. Hence by Propositions 1.3 and 1.4, there exists $g_0 \in G$ such that $Lg_0^{-1}\Gamma/\Gamma$ is dense in G/Γ. Equivalently, $\Gamma g_0 L$ is dense in G. Since L is a normal subgroup of P, for any $x \in P$, $\Gamma g_0 x L = \Gamma g_0 L x$ is dense in G; i.e. $\Gamma(g_0 x)L/L$ is dense in G/L. We now show that there exists $x \in P$ and a sequence $\{\gamma_k\}$ in Γ such that $\gamma_k L \to g_0 x L$ (as elements of G/L), which completes the proof.

Recall that the subgroup H contains $\{\varphi_t\}$ and also U. Hence by Proposition 1.5, every H-orbit on G/Γ is dense. Dualising this as before, we deduce that there exists a sequence γ'_k in Γ such that $\gamma'_k H \to g_0 H$ (as elements of G/H). Hence there exists a sequence $\{h_k\}$ in H such that $\gamma'_k h_k L \to g_0 L$. Let $\sigma \in G$ be an element such that $\sigma \mathbf{v} = -\mathbf{v}$, if the latter condition is satisfied for any element at all; otherwise let σ be the identity element. Then $\sigma \in P$. Since by hypothesis $\varphi_t \mathbf{v} = e^t \mathbf{v}$ we deduce that every element of H is contained in either $\varphi_t \sigma L$ or $\varphi_t L$ for some $t \in \mathbf{R}$. Therefore, by passing to a subsequence we may assume that $h_k = \varphi_{t_k} \sigma_1$ for all $k \in \mathbf{N}$ where σ_1 is either σ or the identity element.

Now recall that $A \cap \Gamma$ is a lattice in A. Since A contains a co-compact abelian Lie subgroup, any lattice in A is co-compact. Hence $A/(A \cap \Gamma)$ is compact. Therefore there exists a compact set C such that $A = (A \cap \Gamma)C$. Let $\{\delta_k\}$ and $\{c_k\}$ be sequences in $A \cap \Gamma$ and C respectively such that $\varphi_{t_k} = \delta_k c_k$. Since C is compact, again by passing to a subsequence if necessary, we may assume that $c_k \to c \in C$. Set $\gamma_k = \gamma'_k \delta_k$. Then

$$\gamma_k L = \gamma'_k \delta_k L = \gamma'_k \varphi_{t_k} c_k^{-1} L = \gamma'_k h_k \sigma_1^{-1} c_k^{-1} L$$
$$= (\gamma'_k h_k L) \sigma_1^{-1} c_k^{-1} \to g_0 L \sigma_1^{-1} c^{-1} = g_0 (\sigma_1^{-1} c^{-1}) L.$$

Note that $c^{-1} \in A \subset P$. Hence $\sigma_1^{-1} c^{-1} \in P$, which completes the proof.

2.4. LEMMA. *Let* $\mathbf{v}$ *be a* p*-frame satisfying condition* (i) *of Theorem* 2.1 *and let* $\mathbf{w} = h\mathbf{v}$ *where* $h \in G$. *Let* Ω *be a given neighbourhood of* h *in* G. *Then there exists* $g \in \Omega$ *and a Cartan subgroup* A *containing* $\{g\varphi_t g^{-1}\}$ *such that* $A\langle g\mathbf{v}\rangle = \langle g\mathbf{v}\rangle$ *and* $A \cap \Gamma$ *is a lattice in* A.

PROOF. Because of condition (i) there exists a Cartan subgroup A' containing $\{\varphi_t\}$ such that $A'\langle \mathbf{v}\rangle = \langle \mathbf{v}\rangle$. By a theorem of G. Prasad and M. S. Raghunathan (cf. [10]) there exists $g \in G$ such that $gA'g^{-1} \cap \Gamma$ is a lattice in $gA'g^{-1}$. Their proof also shows that g can be chosen in any given open subset of G. Then $A = gAg^{-1}$ obviously has the desired properties.

PROOF OF THEOREM 2.1. Let $\mathbf{v} = (v_1, v_2, \cdots, v_p)$ be a p-frame satisfying conditions (i), (ii) and (iii). Let $\mathbf{w}$ be the p-frame as in condition (iii). Since $\lambda_j \gamma_j \mathbf{v} = \gamma_j \varphi_{\log \lambda_j} \mathbf{v} \to \mathbf{w}$ by condition (ii) $\mathbf{w} \in G\mathbf{v}$. Let $h \in G$ be such that $\mathbf{w} = h\mathbf{v}$. Let $v_{p+1}, \cdots, v_n$ be as in condition (i). Set $w_i = hv_i$ for all $i = 1, 2, \cdots, n$. Then $\{w_1, w_2, \cdots, w_n\}$ is a basis of V. Let Ω be a symmetric neighbourhood of the identity in G such that when represented with respect to the basis $\{w_1, w_2, \cdots, w_n\}$ the principal (p, p) minor of any element of Ω is invertible.

By Lemma 2.4 there exists $g \in \Omega h$ and a Cartan subgroup A containing $\{g\varphi_t g^{-1}\}$ such that $A\langle g\mathbf{v}\rangle = \langle g\mathbf{v}\rangle$ and $A \cap \Gamma$ is a (uniform) lattice in A. Set $\mathbf{f} = g\mathbf{v}$. Then by Lemma 2.3, $\Gamma\mathbf{f}$ is dense in $G\mathbf{f}$. We shall show that the closure of $\Gamma\mathbf{v}$

contains a p-frame ξ such that $\langle\xi\rangle = \langle\mathbf{f}\rangle$. In view of condition (ii) and Lemma 2.2 this would complete the proof.

Set $\psi_t = g\varphi_t g^{-1}$ for all $t \in \mathbf{R}$ and $f_i = gv_i$ for all $i = 1, 2, \cdots, n$. Then $\{f_1, f_2, \cdots, f_n\}$ is a basis of V and $\mathbf{f} = (f_1, f_2, \cdots, f_p)$. Also note that for all $t \in \mathbf{R}$, $\psi_t \mathbf{f} = e^t \mathbf{f}$ and $\psi_t f_i = e^{s_i t} f_i$ where $s_i < 1$ for $i \geqq p + 1$. Now let $\{\gamma_j\}$ and $\{\lambda_j\}$ be sequences as given by condition (iii). Put $a_j = \psi_{t_j}$ where $t_j = \log \lambda_j$. Let

$$\gamma_j v_i = \sum_{1 \leqq k \leqq n} \alpha_{i,j}^{(k)} f_k \quad \text{and} \quad w_i = \sum_{1 \leqq k \leqq n} \beta_i^{(k)} f_k$$

be the decompositions with respect to the basis $\{f_1, f_2, \cdots, f_n\}$, where $i = 1, 2, \cdots, p$ and $j \in \mathbf{N}$. Since as $j \to \infty$, $\lambda_j \gamma_j v_i \to w_i$, we obtain that $\lambda_j \alpha_{i,j}^{(k)} \to \beta_i^{(k)}$ for all $i = 1, 2, \cdots, p$ and $k = 1, 2, \cdots, n$. Now

$$\begin{aligned} a_j \gamma_j v_i &= \sum_{1 \leqq k \leqq p} \lambda_j \alpha_{i,j}^{(k)} f_k + \sum_{p+1 \leqq k \leqq n} \lambda_j^{s_k} \alpha_{i,j}^{(k)} f_k \\ &= \sum_{1 \leqq k \leqq p} \lambda_j \alpha_{i,j}^{(k)} f_k + \sum_{p+1 \leqq k \leqq n} \lambda_j^{(s_k - 1)} (\lambda_j \alpha_{i,j}^{(k)}) f_k \\ &\to \sum_{1 \leqq k \leqq p} \beta_i^{(k)} f_k + \sum_{p+1 \leqq k \leqq n} 0 \,.\, \beta_i^{(k)} f_k = \sum_{1 \leqq k \leqq p} \beta_i^{(k)} f_k \end{aligned}$$

as $j \to \infty$. Here we have used that $\lambda_j^{(s_k - 1)} \to 0$ (as $j \to \infty$) which is true since $\lambda_j \to \infty$ and $s_k < 1$ for $k \geqq p + 1$.

Now let C be a compact set such that $A = C(A \cap \Gamma)$. Let $\{c_j\}$ and $\{\gamma'_j\}$ be sequences in C and $A \cap \Gamma$ respectively such that $a_j = c_j \gamma'_j$. By passing to a subsequence if necessary we may assume that $c_j \to c \in C$. Set $\delta_j = \gamma'_j \gamma_j \in \Gamma$. Then for $1 \leqq i \leqq p$, $\delta_j v_i = c_j^{-1} a_j \gamma_j v_i \to c^{-1}(\Sigma \beta_i^{(k)} f_k)$. Notice that in view of our choice of $\mathbf{f} = g\mathbf{v}$, $(\beta_i^{(k)})_{i,k=1,2,\cdots,p}$ is the principal (p,p) minor of a matrix in Ω. Hence $(\beta_i^{(k)})$ is invertible. Set $\xi_i = c^{-1}(\Sigma \beta_i^{(k)} f_k)$ for $i = 1, 2, \cdots, p$. Then $\xi = (\xi_1, \xi_2, \cdots, \xi_p)$ is a p-frame and $\langle\xi\rangle = \langle\mathbf{f}\rangle$. By condition (ii) $\xi \in G\mathbf{v}$ and by choice, $\mathbf{f} \in G\mathbf{v}$. Since $\delta_j \mathbf{v} \to \xi$ and $\delta_j \in \Gamma$ for all j, as observed earlier, by Lemmas 2.2 and 2.3 $\Gamma\mathbf{v}$ is dense in $G\mathbf{v}$.

§3. Orbits under SL(n, Z) and Sp($2n$, Z)

In this section we identify the p-frames satisfying condition (iii) of Theorem 2.1 for the groups $SL(n, \mathbf{Z})$ and $Sp(2n, \mathbf{Z})$ (for $p \leqq n$ in the latter case) and deduce a necessary and sufficient condition for a $SL(n, \mathbf{Z})$ (or $Sp(2n, \mathbf{Z})$) orbit of the frame to be dense in the space of all (resp. all symplectic) p-frames, with p as above.

Let E_r denote the r-rowed identity matrix and 0 the zero matrix of appropriate size. For any matrix $P = (p_{ij})$, we denote $\max_{i,j} |p_{ij}|$ by $\|P\|$, the norm of P.

If $L = (l_{ij})$ is a real (m, p) matrix of rank p such that the only rational column matrix which is a linear combination with coefficients in $\mathbf{R}$, of the p columns of L is the zero column, then the p linear forms $\Sigma_{1 \leqq i \leqq m} l_{ij} x_i$ in $x_1, \cdots, x_m$ for $1 \leqq j \leqq p$ have *rationality rank* p in the sense of Kronecker ([9], §41). Such a matrix L is called *irrational* and any p-frame $\mathrm{v} = (v_1, \cdots, v_p)$ over $\mathbf{R}^m$ for which the matrix $(v_1, \cdots, v_p)$ is irrational is called an *irrational p-frame* over $\mathbf{R}^m$. Observe that a p-frame v is irrational if and only if $\langle \mathrm{v} \rangle$ contains no non-zero rational vector.

3.1. Lemma. *For any integral (m, n) matrix X of rank $m \leqq n$, there exists U in $\mathrm{SL}(n, \mathbf{Z})$ such that $XU = (D, 0)$ with an integral (m, m) matrix D whose inverse has entries bounded by 1 in absolute value.*

Proof. If ${}^t x$ is the first row of X, there exists U_0 in $\mathrm{SL}(n, \mathbf{Z})$ such that ${}^t x U_0 = (d_0, 0, \cdots, 0)$. By induction, there exists U_1 in $\mathrm{SL}(n-1, \mathbf{Z})$ such that if X_1 is the $(m-1, n-1)$ matrix formed by the last $n-1$ columns of the last $m-1$ rows of XU_0, then $X_1 U_1 = (D_1, 0)$ with an invertible integral $(m-1, m-1)$ matrix D_1 for which $\|D_1^{-1}\| \leqq 1$. If now

$$V = U_0 \begin{pmatrix} 1 & 0 \\ 0 & U_1 \end{pmatrix} \quad \text{and} \quad XV = \begin{pmatrix} d_0 & 0 & 0 \\ y & D_1 & 0 \end{pmatrix},$$

there exists an $(m-1)$-rowed integral column t such that $\|D_1^{-1} y + t\| \leqq 1$. Let

$$V_1 = \begin{pmatrix} 1 & 0 \\ t & E_{m-1} \end{pmatrix} \quad \text{and} \quad U = V \begin{pmatrix} V_1 & 0 \\ 0 & E_{n-m} \end{pmatrix};$$

then $XU = (D, 0)$ with

$$D = \begin{pmatrix} d_0 & 0 \\ z & D_1 \end{pmatrix} \quad \text{and} \quad z = y + D_1 t.$$

Since

$$D^{-1} = \begin{pmatrix} d_0^{-1} & 0 \\ -d_0^{-1} D_1^{-1} z & D_1^{-1} \end{pmatrix},$$

it is clear that $\|D^{-1}\| \leqq 1$.

3.2. Proposition. *Given an irrational $(n, n-1)$ real matrix L, there exist a sequence $\{\lambda_t\}$ in $\mathbf{R}_+^*$ tending to ∞ and a sequence $\{M_t\}$ in $\mathrm{SL}(n, \mathbf{Z})$ such that $\lim_{t \to \infty} \lambda_t M_t L$ exists and is of rank $n-1$.*

PROOF. From ([9], satz 62), we know that, since L is irrational, for a given $\varepsilon > 0$ there exists an integral $(n-1, n)$ matrix X such that XL is invertible and further $\|XL\| < \varepsilon$. By Lemma 3.1, we have $X = (D, 0)U^{-1}$ for some U in $\mathrm{SL}(n, \mathbf{Z})$ and for an integral $(n-1, n-1)$ invertible matrix D with $\|D^{-1}\| \leqq 1$. If we set $P = (E_{n-1}, 0)U^{-1}$, then PL is invertible and, moreover, we have $\|PL\| = \|D^{-1}XL\| < (n-1)\varepsilon$. Thus, for

$$U^{-1}L = \begin{pmatrix} PL \\ {}^t\boldsymbol{q} \end{pmatrix},$$

we may now choose an integral column $\boldsymbol{g}$ such that if we set ${}^t\boldsymbol{h} = {}^t\boldsymbol{q}(PL)^{-1} - {}^t\boldsymbol{g}$ then $\|\boldsymbol{h}\| \leqq 1$. Putting

$$V = \begin{pmatrix} E_{n-1} & 0 \\ -{}^t\boldsymbol{g} & 1 \end{pmatrix} U^{-1},$$

we see that $\|VL\| < (n-1)^2\varepsilon$. It is immediate then that a sequence $\{\lambda_t\}$ in $\mathbf{R}_+^*$ tending to ∞ and a sequence $\{M_t\}$ in $\mathrm{SL}(n, \mathbf{Z})$ exist such that the $(n, n-1)$ real matrix $A = \lim_{t \to \infty} \lambda_t M_t L$ has rank at least equal to 1.

Let

$$A = \begin{pmatrix} E_r & 0 \\ * & 0 \end{pmatrix} B \qquad \text{with } r \geqq 1,$$

for an invertible $(n-1, n-1)$ real matrix B. If $r < n-1$, we will show that M_t can be suitably modified in $\mathrm{SL}(n, \mathbf{Z})$ so as to obtain the same assertion about A with $r+1$ instead of r. This will clearly complete the proof of Proposition 3.2 by induction.

From above, we have for all sufficiently large t,

$$\lambda_t M_t L = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} B$$

with A_1 close to E_r and A_2, A_4 close to 0. Evidently, there exists a sequence $\{C_t\}$ contained in a fixed compact neighbourhood Ω of E_{n-1} such that

$$\lambda_t M_t L = \begin{pmatrix} E_r & 0 \\ a_1 \cdots a_r & a_{r+1} \cdots a_{n-1} \\ b_1 \cdots b_r & b_{r+1} \cdots b_{n-1} \\ * & * \end{pmatrix} C_t B.$$

Since PL has rank $n-1$, we may, without loss of generality, suppose that not all b_j, for $j > r$, are zero. Let $|b_s| = \max_{j>r} |b_j|$; we may assume that $s = r+1$, after suitably modifying Ω. Now there exists k (depending on t) in $\mathbf{Z}$ such that $1 \leqq |kb_s + a_s| \leqq 2$. Further, for $j > r$, we have

$$|kb_j + a_j| \leqq |a_j| + |k||b_s| \leqq |a_j| + 2 + |a_s|.$$

Since, for $j > r$, all a_j and b_j tend to 0 as t tends to infinity, it is clear that $kb_j + a_j$ are bounded for $j > r$. Choose $n_1, \cdots, n_r$ in $\mathbf{Z}$ such that $|kb_i + a_i + n_i| < 1$ for $1 \leqq i \leqq r$. Setting

$$M_t^* = \begin{pmatrix} E_r & & 0 \\ n_1 \cdots n_r & 1\,k & 0 \cdots 0 \\ 0 \cdots 0 & 0\,1 & 0 \cdots 0 \\ 0 & 0 & E_{r-2} \end{pmatrix} M_t,$$

it is easy to check that, for a suitable subsequence $\{\lambda_l\}$ of $\{\lambda_t\}$, $\lim_{l\to\infty} \lambda_l M_l^* L$ exists and has rank at least equal to $r+1$ and we are through.

We now take up the analogue of Proposition 3.2 for the case of $\mathrm{Sp}(2n, \mathbf{Z})$.

3.3. Proposition. *Let*

$$L = \begin{pmatrix} P \\ Q \end{pmatrix}$$

be a real $(2n, n)$ *matrix of rank* n *with* ${}^tPQ = {}^tQP$ *and let* L *be irrational. Then there exists a sequence* $\{\lambda_t\}$ *in* $\mathbf{R}_+^*$ *tending to* ∞ *and a sequence* $\{M_t\}$ *in* $\mathrm{Sp}(2n, \mathbf{Z})$ *such that* $\lim_{t\to\infty} \lambda_t M_t L$ *exists and has rank* n.

Proof. Assume that $1 \leqq r < n$ and that, for any $\varepsilon > 0$, there exists an integral $(r, 2n)$ matrix X for which XL has rank r, $\|XL\| < \varepsilon$ and further,

$$X \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} {}^tX = 0.$$

We shall then uphold the same assertion with $r+1$ ($\leqq n$) instead of r. For $r = 1$, the existence of such an X follows (cf. [9]) from the fact that the n linear forms (in $2n$ variables) corresponding to L have positive rationality rank. If

$$X^* = X \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix},$$

then the rationality rank corresponding to $(L, {}^tX^*)$ is still n. Therefore, from [9], there exists an integral $(n, 2n)$ matrix Y such that $\|Y(L, {}^tX^*)\| < \varepsilon$ and further $Y(L, {}^tX^*)$ has rank n. Since we may certainly suppose that $\varepsilon < 1$, we have $Y(L, {}^tX^*) = (YL, 0)$ and therefore YL has rank n and further $\|YL\| < \varepsilon$. If ${}^ty_1, \cdots, {}^ty_n$ are the rows of Y, we assert that at least one of $y_1, \cdots, y_n$, say y, has the property that

$$\begin{pmatrix} X \\ {}^ty \end{pmatrix} L$$

has rank $r+1$; for, the number of linearly independent rows of the form ${}^t xL$ with

$$\begin{pmatrix} X \\ {}^t x \end{pmatrix} L$$

having rank r is at most equal to r, while we know that YL has rank n. Continuing the induction argument up to $r = n-1$, we obtain, for $\varepsilon > 0$, an integral $(n, 2n)$ matrix X such that XL has rank n, $\|XL\| < \varepsilon$ and further

$$X \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} {}^t X = 0.$$

Now by elementary divisor theorem and ([11], p. 101) there exists an invertible integral (n, n) matrix D such that $X = D\tilde{X}$ and $\tilde{X}$ can be completed to an element M^* of $\mathrm{Sp}(2n, \mathbf{Z})$. Replacing $\tilde{X}$ and D by $U\tilde{X}$ and DU^{-1} respectively, with a suitable U in $\mathrm{SL}(n, \mathbf{Z})$, we may, in view of Lemma 3.1, suppose already that $\|D^{-1}\| \leqq 1$. Thus, for any $\varepsilon > 0$, we have shown that there exists an integral $(n, 2n)$ matrix $\tilde{X}$ with some

$$M^* = \begin{pmatrix} \tilde{X} \\ * \end{pmatrix}$$

in $\mathrm{Sp}(2n, \mathbf{Z})$ such that $P_1 = \tilde{X}L$ has rank n and $\|P_1\| < \varepsilon$. If

$$M^*L = \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix},$$

then $Q_1 P_1^{-1} = {}^t(Q_1 P_1^{-1}) = T + R$ for an integral symmetric T chosen such that $\|R\| < 1$. Then

$$\left\| \begin{pmatrix} E_n & 0 \\ -T & E_n \end{pmatrix} M^*L \right\| < n\varepsilon$$

and we are in a position to conclude the existence of $\{\lambda_t\}$ tending to ∞ and $\{M_t\}$ in $\mathrm{Sp}(2n, \mathbf{Z})$ such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \lim_{t \to \infty} \lambda_t M_t L$$

exists and rank $A \geqq 1$.

We next show that if $\{M_t\}$ and $\{\lambda_t\}$ are sequences in $\mathrm{Sp}(2n, \mathbf{Z})$ and $\mathbf{R}_+^*$ respectively, such that $\lambda_t \to \infty$ and

$$\lim_{t \to \infty} \lambda_t M_t L = \begin{pmatrix} A \\ B \end{pmatrix}$$

where A and B are (n, n) matrices and A has rank p, $1 \leqq p < n$, then there exist sequences $\{M'_t\}$ and $\{\lambda'_t\}$ in $\mathrm{Sp}(2n, \mathbf{Z})$ and $\mathbf{R}^*_+$ respectively such that $\lambda'_t \to \infty$ and

$$\lim_{t\to\infty} \lambda'_t M'_t L = \begin{pmatrix} A' \\ B' \end{pmatrix}$$

where A' and B' are (n, n) matrices and A' has rank at least $p + 1$. The proposition then follows from an obvious induction argument.

Let the notation be as above and let $C \in \mathrm{GL}(n, \mathbf{R})$ be such that

$$A = \begin{pmatrix} E_p & 0 \\ A_1 & 0 \end{pmatrix} C.$$

As before there exists a bounded sequence $\{C_t\}$ in $\mathrm{GL}(n, \mathbf{R})$ such that $\lambda_t M_t L = V_t C_t C$ where V_t is a bounded sequence of matrices having the form

$$V_t = \begin{pmatrix} E_p & 0 \\ X_t & X'_t \\ L_t & L'_t \\ Y_t & Y'_t \end{pmatrix}$$

where X_t and Y_t are $(n - p, p)$ matrices, X'_t and Y'_t are $(n - p, n - p)$ matrices, L_t is a (p, p) matrix, L'_t is a $(p, n - p)$ matrix and 0 is the $(p, n - p)$ zero matrix. If either $\{X'_t\}$ or $\{Y'_t\}$ does not tend to zero then our assertion about improving the rank is obviously satisfied. Hence we may assume that $X'_t \to 0$ and $Y'_t \to 0$. Observe that since $M_t L$ has rank n, for any t,

$$\begin{pmatrix} X'_t \\ L'_t \\ Y'_t \end{pmatrix} \neq 0.$$

Replacing M_t by γM_t where $\gamma \in \mathrm{Sp}(2n, \mathbf{Z})$ (and $C_t C$ by another bounded sequence of matrices) we may assume that for t in a subsequence, $Y'_t \neq 0$. If X'_t is non-zero for a subsequence in t then this can be accomplished by a permutation matrix. On the other hand if X'_t and Y'_t are zero for all large t and l is such that the lth row of L'_t is non-zero for a subsequence in t we may choose

$$\gamma = \left(\begin{array}{cc|cc} E_p & -{}^tT & \multicolumn{2}{c}{0} \\ 0 & E_{n-p} & & \\ \hline \multicolumn{2}{c|}{0} & E_p & 0 \\ & & T & E_{n-p} \end{array}\right)$$

where T is the matrix whose entries other than the $(1, l)$th are zero and the

$(1, l)$th entry is $m \in \mathbf{Z} - (0)$. Passing to a subsequence if necessary, we may therefore assume that there exists i, $1 \leqq i \leqq n - p$ such that $\boldsymbol{b}'_t$, the ith row of Y'_t, is non-zero for all t. Let $\boldsymbol{a}_t$, $\boldsymbol{b}_t$ and $\boldsymbol{a}'_t$ denote the ith rows of X_t, Y_t and X'_t respectively (in the corresponding subsequence). Since $\boldsymbol{b}'_t \neq 0$ and $\boldsymbol{a}'_t$ and $\boldsymbol{b}'_t$ tend to zero, for sufficiently large t there exist $k_t \in \mathbf{Z}$ such that

(a) $1 \leqq \| \boldsymbol{a}'_t + k_t \boldsymbol{b}'_t \| \leqq 2$.

For all large t, let $\boldsymbol{m}_t$ and $\boldsymbol{v}_t$ be $(1, p)$ matrices such that $\boldsymbol{m}_t$ is an integral matrix, $\| \boldsymbol{v}_t \| < 1$ and

(b) $\boldsymbol{a}_t + k_t \boldsymbol{b}_t = \boldsymbol{m}_t + \boldsymbol{v}_t$.

Let Z_t be the $(n-p, p)$ matrix whose ith row is $\boldsymbol{m}_t$ and all others are zero. Finally, for all large t let T_t be an integral symmetric matrix such that

(c) $\| \frac{1}{2}({}^t\boldsymbol{m}_t \boldsymbol{b}_t + {}^t\boldsymbol{b}_t \boldsymbol{m}_t) + T_t \| < 1$.

Consider the elements $\sigma_t \in \mathrm{Sp}(2n, \mathbf{Z})$ defined (for all large t) by

$$\sigma_t = \left(\begin{array}{c|c} E_n & 0 \\ \hline \begin{matrix} T_t & 0 \\ 0 & 0 \end{matrix} & E_n \end{array}\right) \left(\begin{array}{c|c} \begin{matrix} E_p & 0 \\ -Z_t & E_{n-p} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} E_p & {}^tZ_t \\ 0 & E_{n-p} \end{matrix} \end{array}\right) \left(\begin{array}{c|c} E_n & k_t B_i \\ \hline 0 & E_n \end{array}\right)$$

B_i being the (n, n) matrix whose $(p+i, p+i)$th entry is 1 and all others are zero. We show that $\lambda_t \sigma_t M_t L$ is a bounded sequence of matrices and that if

$$\begin{pmatrix} A' \\ B' \end{pmatrix}$$

is the limit of a convergent subsequence where A' and B' are (n, n) matrices then A' has rank at least $p + 1$. Let

$$\sigma_t V_t = \begin{pmatrix} K_t \\ R_t \end{pmatrix}$$

where K_t and R_t are (n, n) matrices. Then clearly

$$K_t = \begin{pmatrix} E_p & 0 \\ -Z_t + W_t & W'_t \end{pmatrix}$$

where W_t and W'_t are identical to X_t and X'_t respectively, except for the ith rows, the latter being $\boldsymbol{a}_t + k_t \boldsymbol{b}_t$ and $\boldsymbol{a}'_t + k_t \boldsymbol{b}'_t$ respectively. Since V_t is a bounded sequence, in view of (a) and (b) it follows that K_t is bounded. Again clearly

$$R_t = \begin{pmatrix} T_t + L_t + {}^tZ_t Y_t & L'_t + {}^tZ_t Y'_t \\ Y_t & Y'_t \end{pmatrix}.$$

But

$$T_t + {}^tZ_tY_t = T_t + {}^t\boldsymbol{m}_t\boldsymbol{b}_t$$
$$= \{T_t + \tfrac{1}{2}({}^t\boldsymbol{m}_t\boldsymbol{b}_t + {}^t\boldsymbol{b}_t\boldsymbol{m}_t)\} + \tfrac{1}{2}({}^t\boldsymbol{m}_t\boldsymbol{b}_t - {}^t\boldsymbol{b}_t\boldsymbol{m}_t).$$

By (c) the first term is bounded. In view of (b) the second term equals $\frac{1}{2}\{({}^t\boldsymbol{a}_t - {}^t\boldsymbol{v}_t)\boldsymbol{b}_t - {}^t\boldsymbol{b}_t(\boldsymbol{a}_t - \boldsymbol{v}_t)\}$ and the latter is bounded since V_t and $\boldsymbol{v}_t$ are bounded. Also

$${}^tZ_tY'_t = {}^t\boldsymbol{m}_t\boldsymbol{b}'_t = ({}^t\boldsymbol{a}_t + k_t{}^t\boldsymbol{b}_t - {}^t\boldsymbol{v}_t)\boldsymbol{b}'_t$$
$$= ({}^t\boldsymbol{a}_t - {}^t\boldsymbol{v}_t)\boldsymbol{b}'_t + {}^t\boldsymbol{b}_t(k_t\boldsymbol{b}'_t).$$

In view of (a), $k_t\boldsymbol{b}'_t$ is bounded. Since the other terms are also bounded as observed earlier, we conclude that ${}^tZ_tY'_t$ is bounded. Since Y_t and Y'_t are bounded it now follows that R_t is bounded. Therefore

$$\lambda_t\sigma_tM_tL = (\sigma_tV_t)C_tC = \begin{pmatrix} K_t \\ R_t \end{pmatrix} C_tC$$

is a bounded sequence of matrices.

Finally let

$$\begin{pmatrix} A' \\ B' \end{pmatrix},$$

where A' and B' are (n, n) matrices, be a limit of a subsequence of $\{\lambda_t\sigma_tM_tL\}$. Then $A' = K'C'C$ where K' and C' are limits of certain subsequence of $\{K_t\}$ and $\{C_t\}$ respectively. Since

$$K_t = \begin{pmatrix} E_p & 0 \\ -Z_t + W_t & W'_t \end{pmatrix}$$

and by (a) W' has no subsequence tending to zero, it follows that K' has rank at least $p + 1$. Since $C'C$ is non-singular $A' = K'C'C$ has rank at least $p + 1$.

3.4. THEOREM. (i) *Let* $G = \mathrm{SL}(n, \mathbf{R})$, $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ *and* $\mathbf{v}$ *any* p*-frame over* $\mathbf{R}^n$ *with* $p \leqq n - 1$. *Then* $\Gamma\mathbf{v}$ *is dense in the space of all* p*-frames over* $\mathbf{R}^n$ (*and hence in* $(\mathbf{R}^n)^p$) *if and only if* $\mathbf{v}$ *is irrational.*

(ii) *Let* $G = \mathrm{Sp}(2n\ \mathbf{R})$, $\Gamma = \mathrm{Sp}(2n, \mathbf{Z})$ *and* $\mathbf{v}$ *any symplectic* p*-frame over* $\mathbf{R}^{2n}$ *with* $p \leqq n$. *Then* $\Gamma\mathbf{v}$ *is dense in the space of all symplectic* p*-frames over* $\mathbf{R}^{2n}$ *if and only if* $\mathbf{v}$ *is irrational.*

NOTE. It turns out that in the case of $\mathrm{Sp}(2n, \mathbf{Z})$ Theorem 3.4 can be deduced from Proposition 3.3 using Proposition 1.5, without involving Theorem 2.1. For the case of $\mathrm{SL}(n, \mathbf{Z})$ however, that method works only for $p \leqq n/2$. We include only the following general proof.

PROOF. Clearly, it is enough to prove the theorem for $p = n-1$ and n in cases (i) and (ii) respectively. If $\mathbf{v}$ is an irrational p-frame satisfying the hypotheses (in either case), then in view of the foregoing the conditions of Theorem 2.1 are fulfilled and therefore $\Gamma\mathbf{v}$ is dense in $G\mathbf{v}$, which is just the space of all p-frames over $\mathbf{R}^n$ in case (i) and the space of all symplectic p-frames over $\mathbf{R}^{2n}$ in case (ii).

Let $\mathbf{v} = (v_1, \cdots, v_p)$ be a p-frame which is not irrational. Then there exists a p-frame $\mathbf{w} = (w_1, \cdots, w_p)$ with w_p integral such that $\langle \mathbf{v} \rangle = \langle \mathbf{w} \rangle$. Clearly $\Gamma\mathbf{w}$ cannot be dense in $G\mathbf{w}$ and therefore by Lemma 2.2, $\Gamma\mathbf{v}$ cannot be dense in $G\mathbf{v}$.

§4. Application to horospherical flows

We now apply Theorem 3.5 to study density of orbits of certain horospherical subgroups of $\mathrm{SL}(n, \mathbf{R})$. (cf. §1, (b) for the definition and recollection of certain well-known properties).

4.1. COROLLARY. *Let U be a (minimal) horospherical subgroup of $G = \mathrm{SL}(n, \mathbf{R})$ such that U is the unipotent radical of a maximum dimensional parabolic subgroup of G. Let $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ and $x \in G$. Then the following statements are equivalent*:

(i) *The U-orbit $Ux\Gamma/\Gamma$ is not dense in G/Γ.*

(ii) *$x^{-1}Ux$ is contained in a maximum dimensional parabolic subgroup defined over $\mathbf{Q}$.*

(iii) *$x^{-1}Ux$ is contained in a proper $\mathbf{R}$-algebraic subgroup which is defined over $\mathbf{Q}$.*

PROOF. It is well-known that if P is a maximum dimensional parabolic subgroup of $\mathrm{SL}(n, \mathbf{R})$ then either P or ${}^tP = \{{}^tA \mid A \in P\}$ is the isotropy subgroup of an element in $\mathbf{P}^{n-1}$ under the natural action. Evidently it is enough to prove the Corollary in the latter case. Observe also that the Corollary is true for U if and only if it is true for some conjugate gUg^{-1}, $g \in G$. Let $\{e_i, i = 1, 2, \cdots, n\}$ be the standard basis of $\mathbf{R}^n$ and $\pi : \mathbf{R}^n - (0) \to \mathbf{P}^{n-1}$ be the natural projection. Since the action of G on $\mathbf{P}^{n-1}$ is transitive by replacing U by an appropriate conjugate if necessary, we may assume ${}^tP$ to be the isotropy subgroup of $\pi(e_n)$. Then it is easy to verify that U is the isotropy subgroup of the $(n-1)$-frame $\mathbf{e} = (e_1, e_2, \cdots, e_{n-1})$. We now prove the equivalence of the statements for this U.

(i) $\Rightarrow$ (ii). Suppose that $Ux\Gamma/\Gamma$ is not dense in G/Γ. Then $Ux\Gamma$ is not dense in G and hence $\Gamma x^{-1}U$ is not dense in G. Hence by the remarks following the statement of Theorem 2.1, if $\mathbf{v} = x^{-1}\mathbf{e}$ then $\Gamma\mathbf{v}$ is not dense in $G\mathbf{v}$. Therefore by Theorem 3.5, $\mathbf{v}$ is not irrational. In other words, there exists a $(n-1)$-frame $\mathbf{w} = (w_1, \cdots, w_{n-1})$ such that w_1 is an integral vector and $\langle \mathbf{v}\rangle = \langle \mathbf{w}\rangle$. Let Q be the isotropy subgroup of $\pi(w_1)$ under the natural action. Then Q is a maximum dimensional parabolic subgroup defined over $\mathbf{Q}$. Also since $w_1 \in \langle \mathbf{v}\rangle$, $x^{-1}Ux$ is contained in Q.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Let L be a proper $\mathbf{R}$-algebraic subgroup defined over $\mathbf{Q}$ and containing $x^{-1}Ux$. Then there exists a $\mathbf{Q}$-rational representation $\rho : G \to GL(\mathbf{R}^q)$ for some $q > 1$ and a rational vector $w_0 \in \mathbf{R}^q$ such that

$$L \subset \{g \in G \mid \rho(g)w_0 = tw_0 \text{ for some } t \in \mathbf{R}\}$$

(cf. proposition 7.8 [2]). Also set

$$L' = \{g \in G \mid \rho(g)w_0 = w_0\}.$$

Since $x^{-1}Ux$ consists of unipotent elements it is contained in L'. Since w_0 is a rational vector $\rho(\Gamma)w_0$ is a closed subset of $\mathbf{R}^q$. Therefore $\Gamma L'$ is a closed subset of G. Evidently it is also proper. Hence $\Gamma x^{-1}U \subset \Gamma L' x^{-1}$ is not dense in G. Therefore by inverting, as before we conclude that $Ux\Gamma/\Gamma$ is not dense in G/Γ.

4.2. Proposition. *Let $\mathbf{v} = (v_1, v_2, \cdots, v_{n-1})$ be a $(n-1)$-frame over $\mathbf{R}^n$ such that $(v_1, v_2, \cdots, v_p)$, where $1 \leqq p < n-1$, is an irrational p-frame and $v_{p+1}, v_{p+2}, \cdots, v_{n-1}$ are rational vectors. Let*

$$\Gamma_0 = \{\gamma \in SL(n, \mathbf{Z}) \mid \gamma v_i = v_i \text{ for } i \geqq p+1\}.$$

Then $\Gamma_0(v_1, v_2, \cdots, v_p)$ is dense in the space of p-frames over $\mathbf{R}^n$ (and hence in $(\mathbf{R}^n)^p$).

Proof. Replacing $\mathbf{v}$ by $\mathbf{v}A$ where $A \in GL(n-1, \mathbf{R})$ is chosen suitably, we may assume that $(v_{p+1}, v_{p+2}, \cdots, v_{n-1})$, viewed as a $(n, n-p-1)$ matrix, is integral and primitive. Then there exists $\gamma \in SL(n, \mathbf{Z})$ such that

$$\gamma(v_{p+1}, v_{p+2}, \cdots, v_{n-1}) = (e_{p+2}, e_{p+3}, \cdots, e_n).$$

Therefore again replacing $\mathbf{v}$ by $\gamma\mathbf{v}$ we may assume without loss of generality that $(v_{p+1}, \cdots, v_{n-1}) = (e_{p+2}, \cdots, e_n)$. Then $\mathbf{v}$ viewed as a matrix has the form

$$\begin{pmatrix} C & 0 \\ D & E_{n-p-1} \end{pmatrix}$$

where C and D are $(p+1,p)$ and $(n-p-1,p)$ matrices respectively. Further, the columns of C form an irrational p-frame over $\mathbf{R}^{p+1}$. Also with the above assumptions,

$$\Gamma_0 = \left\{ \begin{pmatrix} X & 0 \\ Y & E_{n-p-1} \end{pmatrix} \middle| \begin{matrix} X \in \mathrm{SL}(p+1,\mathbf{Z}) \text{ and } Y \text{ is any} \\ \text{integral } (n-p-1,p+1) \text{ matrix} \end{matrix} \right\}.$$

Now let

$$\begin{pmatrix} R \\ S \end{pmatrix}$$

be the matrix corresponding to a p-frame over $\mathbf{R}^n$ where R and S are $(p+1,p)$ and $(n-p-1,p)$ matrices respectively. Let $\varepsilon > 0$ be arbitrary. Since C is an irrational p-frame there exists $X \in \mathrm{SL}(p+1,\mathbf{Z})$ such that $\|XC\| < \varepsilon$. Let B be an invertible (p,p) minor of XC. Let Y' be an integral $(n-p-1,p)$ matrix such that $\|(D-S)B^{-1} - Y'\| \leqq 1$ and let Y be a $(n-p-1,p+1)$ matrix such that $YXC = Y'B$. Setting

$$\gamma_1 = \begin{pmatrix} E_{p+1} & 0 \\ Y & E_{n-p-1} \end{pmatrix} \cdot \begin{pmatrix} X & 0 \\ 0 & E_{n-p-1} \end{pmatrix}$$

we see that

$$\gamma_1 \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} XC \\ YXC + D \end{pmatrix}$$

Observe that $\|YXC + D - S\| = \|Y'B + D - S\| = \|\{Y' + (D-S)B^{-1}\}B\| < p\varepsilon$. Also by irrationality of XC and Theorem 3.5 we can find $X' \in \mathrm{SL}(p+1,\mathbf{Z})$ such that $\|X'XC - R\| < \varepsilon$. Put

$$\gamma_0 = \begin{pmatrix} X' & 0 \\ 0 & E_{n-p-1} \end{pmatrix} \cdot \gamma_1.$$

Then

$$\gamma_0 \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} X'XC \\ YXC + D \end{pmatrix}$$

and hence

$$\left\| \gamma_0 \begin{pmatrix} C \\ D \end{pmatrix} - \begin{pmatrix} R \\ S \end{pmatrix} \right\| < p\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\gamma_0 \in \Gamma_0$, we conclude that

$$\binom{R}{S} \text{ is in the closure of } \Gamma_0 \binom{C}{D}.$$

Hence $\Gamma_0(v_1, v_2, \cdots, v_p)$ is dense.

4.3. Corollary. *As before, let* $G = \mathrm{SL}(n, \mathbf{R})$, $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ *and* U *the unipotent radical of a maximum dimensional parabolic subgroup of* G. *Then for all* $x \in G$, *the closure of* $Ux\Gamma/\Gamma$ *in* G/Γ *is of the form* $Hx\Gamma/\Gamma$ *where* H *is a (closed) subgroup of* G.

Proof. By the inversion argument used so often, it is enough to prove that for any $y \in G$ the closure of ΓyU is of the form ΓyH where H is a subgroup of G. So let $y \in G$ and put $\mathbf{v} = y\mathbf{e}$ where, as in Corollary 4.1, $\mathbf{e} = (e_1, e_2, \cdots, e_{n-1})$. As in Corollary 4.1 we may assume U to be the isotropy subgroup of $\mathbf{e} = (e_1, e_2, \cdots, e_{n-1})$ in G. Let $A \in \mathrm{GL}(n-1, \mathbf{R})$ be such that $\mathbf{w} = \mathbf{v}A = (w_1, w_2, \cdots, w_{n-1})$ where $(w_1, w_2, \cdots, w_p)$ is an irrational p-frame with $p \leqq n-1$ and $w_{p+1}, \cdots, w_{n-1}$ are rational vectors. If $p = n-1$ then $\mathbf{v}$ is irrational and therefore by Theorem 3.5, ΓyU is dense in G; in this case, we may choose $H = G$. Next let $p < n-1$. Put

$$G' = \{g \in G \mid gw_i = w_i \text{ for all } i \geqq p+1\} \quad \text{and} \quad \Gamma' = \Gamma \cap G'.$$

By Proposition 4.2, $\Gamma'(w_1, w_2, \cdots, w_p)$ is dense in $G'(w_1, w_2, \cdots, w_p)$. Since obviously $\Gamma G'(w_1, w_2, \cdots, w_{n-1})$ is a discrete union of $\{\gamma_k G'(w_1, \cdots, w_{n-1})\}$ where $\{\gamma_k \mid k \in \mathbf{N}\}$ is a set of representatives of Γ/Γ', the above implies that the closure of $\Gamma\mathbf{w}$ in $G\mathbf{w}$ is precisely $\Gamma G'\mathbf{w}$. Hence $\Gamma G'\mathbf{v}$ is the closure of $\Gamma\mathbf{v}$ in $G\mathbf{v}$. Recall that yUy^{-1} is the isotropy subgroup of $\mathbf{v}$ and hence yUy^{-1} is contained in G'. Therefore the above assertion means that $\Gamma(yUy^{-1})$ has $\Gamma G'$ as its closure in G. Hence if we put $H = y^{-1}G'y$, then ΓyH is the closure of ΓyU in G, which proves the Corollary.

Corollary 4.3 signifies that the closure of every orbit is geometrically a nice object — a homogeneous space. In [6] a similar result is proved for minimal (closed invariant) sets of maximal horospherical flows on homogeneous spaces of certain semisimple Lie groups. In a forthcoming paper, the results of [6] are generalised to maximal horospherical flows of any semisimple Lie group on any of its homogeneous spaces with finite invariant volume.

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